

Modeling Non-Axisymmetric Bow Shocks: Solution Method and Exact Analytic Solutions

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ABSTRACT

A new solution method is presented for steady-state, momentum-conserving, non-axisymmetric bow shocks and colliding winds in the thin-shell limit. This is a generalization of previous formulations to include a density gradient in the pre-shock ambient medium, as well as anisotropy in the pre-shock wind. For cases where the wind is unaccelerated, the formalism yields exact, analytic solutions.

Solutions are presented for two bow shock cases: (1) that due to a star moving supersonically with respect to an ambient medium with a density gradient perpendicular to the stellar velocity, and (2) that due to a star with a misaligned, axisymmetric wind moving in a uniform medium. It is also shown under quite general circumstances that the total rate of energy thermalization in the bow shock is independent of the details of the wind asymmetry, including the orientation of the non-axisymmetric driving wind, provided the wind is non-accelerating and point-symmetric. A typical feature of the solutions is that the region near the standoff point is tilted, so that the star does not lie along the bisector of a parabolic fit to the standoff region. The principal use of this work is to infer the origin of bow shock asymmetries, whether due to the wind or ambient medium, or both.

Subject headings: hydrodynamics — ISM: bubbles — ISM: HII regions—shock waves
—stars: mass loss

1. Introduction

Supersonic stellar winds shock the surrounding gas and drive expanding bubbles into the interstellar medium. These shocks provide an opportunity to probe the properties of both the driving stellar wind and the ambient medium. If the star is moving with respect to the interstellar gas, the bubble will be distorted into a cometary shape. When the stellar motion is supersonic, we refer to these as stellar wind bow shocks (Baranov, Krasnobaev & Kulikovskii 1971; Dyson 1975). Since the discovery of such bow shocks around young B stars (Van Buren & McCray 1988), bow shocks have been found associated with many classes of objects, such as pulsars (Kulkarni et al. 1992) and cataclysmic variables (e.g. Vela X-1: Kaper et al. 1997); examples include

well-known naked-eye stars (e.g. Betelgeuse: Noriega-Crespo et al. 1997). Bow shocks have been proposed as an explanation for cometary, ultracompact HII regions (Van Buren et al. 1990; Mac Low et al. 1991) and as a means of explaining the lifetimes of ultracompact HII regions. In a recent survey of the IRAS database using HiRes processing, Van Buren, Noriega-Crespo, & Dgani (1995) found 58 candidate bow shocks.

Non-axisymmetric stellar wind bow shocks occur when a star with an anisotropic wind moves supersonically with respect to the local medium, or if the star has an isotropic wind but moves in an ambient medium containing a transverse density gradient. Models of non-axisymmetric bow shocks are relevant to cometary ultracompact HII regions due to wind-blowing O stars moving supersonically with respect to the surrounding molecular cloud, when the ambient material does not have a constant density. A non-axisymmetric bow shock has also been invoked to explain the morphology of Kepler’s supernova remnant, where the supernova ejecta collide with a non-axisymmetric bow shock generated by the pre-supernova wind (Bandiera 1987; Borkowski, Blondin & Sarazin 1992). Another example is the bow shock due to the head of a jet propagating into a region with a density gradient. Non-axisymmetric, ram-pressure balance models of the collision between a stellar wind and the photoevaporating flow from an externally illuminated circumstellar disk have been given by Henney et al. (1996). A formulation for steady-state non-axisymmetric bow shocks and colliding winds was given by Bandiera (1993). However, Bandiera’s numerical method is sufficiently complicated that a simpler, analytic method is desirable.

In this contribution, I present a method for solving the problem of steady-state, momentum-conserving, non-axisymmetric, thin-shell bow shocks and colliding winds. This is an extension of the previous analytic solution method of Wilkin (1996, hereafter Paper I) and of Cantó, Raga, & Wilkin (1996, hereafter Paper II) to non-axisymmetric problems (see also Wilkin 1997a).

An outline of the paper is as follows. In § 2, we formulate the problem of the steady-state collision of two winds, and in § 3 we treat the problem of a bow shock resulting from an isotropic wind interacting with a plane-parallel flow containing a transverse density gradient. In § 4, we allow for non-isotropic winds, especially an axisymmetric wind with random orientation of the symmetry axis with respect to the direction of stellar motion. The rate at which kinetic energy is thermalized for the bow shock is treated in § 5. Results and future directions of this research are summarized in § 6.

2. Mathematical Formulation and Solution Method

2.1. Description of the Collision Surface

The hypersonic collision of two winds will in general result in a system of two shocks. A specific example, that of a stellar wind bow shock, is shown schematically in Figure 1. Because this

paper is concerned with steady-state solutions, the colliding winds are assumed to be unchanging in time. The stellar wind bow shock arises when a wind-blowing star moves supersonically with respect to the intersellar gas. In this case, we formulate the problem in the reference frame of the star, so the ambient medium is described as a wind of parallel streamlines impinging on the bow shock. For the collision of two winds in a binary star, we will neglect orbital motion in order to consider a steady-state problem in an inertial reference frame. In steady-state, the amount of mass and momentum within a given volume does not increase, and a flow pattern exists between the two shocks that carries away the mass and momentum deposited by the colliding winds. If the shocks are radiative, post-shock cooling lowers the temperature of the gas and leads to a large compression. We make the strong thin-shell assumption that cooling is so efficient that the shocked shell collapses to an infinitesimally thin layer, with a finite surface density σ of matter. For this thin-shell assumption to apply, it is also necessary that the post-shock gas not be supported by magnetic fields, which, if well-coupled to the gas, could maintain a finite thickness even in the presence of efficient cooling. We assume there are no other forces such as those due to radiation or magnetic fields. The two shocked winds may in principle be separated by a contact discontinuity. However, there would then be supersonic shear across the interface that is expected to be unstable, leading to a mixing of the two fluids. A detailed treatment of the mixing is beyond the scope of this work. Instead, the mixing is assumed to be instantaneous, so the shell will have a unique velocity \mathbf{V}_t at any location within it. This velocity represents an average of the turbulent fluctuations that would be present in a more detailed treatment. In steady-state, the geometric shape of the shell is unchanging in time, so the velocity of matter within the infinitely thin layer must be purely tangential to the shell. There will, however, be acceleration normal to the shell, because the fluid typically does not follow a straight path. The assumption that the incident streams are hypersonic, combined with the perfect cooling assumption, means that the flow within the shell will also be hypersonic, and pressure forces may be neglected in describing the motions along the shell.¹ Defining a spherical coordinate system (r, θ, ϕ) , and denoting the radius of the surface by $r = R(\theta, \phi)$, a complete description of this idealized shell is given by specifying the quantities R, σ, \mathbf{V}_t as a function of position (θ, ϕ) within the shell. There are in fact only four independent quantities, because the condition that the motion within the shell be tangential implies that one need only solve for two velocity components.

¹Clearly this assumption must break down for real, finite temperature systems near the stagnation point, because the tangential flow velocity in the shell vanishes at that point. However, pressure forces depend upon the gradient of the pressure, which also vanishes at the stagnation point, so the pressure forces may still be small compared to momentum deposition in the stagnation region (Wilkin 1997b). In any case, beyond a small region near the stagnation point, the tangential flow in the shell will be supersonic, provided the incident flows are supersonic and the shocked fluid cools efficiently.

2.2. Previous Treatment of the Problem

The conservation laws of mass and momentum may be used to derive the properties of the shell in terms of those of the two incident winds. Since the momentum conservation law has three components, there will be four equations in four unknowns, or including the condition of zero normal velocity, five equations in five unknowns. Because the assumption of vanishing thickness eliminates one spatial variable, these equations will be partial differential equations (PDEs) in two spatial coordinates. Suitable boundary conditions must also be specified. In practice, this means identifying the stagnation (or standoff) point, where the two winds collide head-on, and where for steady-state, the ram pressures of the two fluids balance. The equations may then be integrated away from the standoff point, following the motion of a fluid element. In order to begin the numerical integrations, one generally finds that an expansion about the conditions at the stagnation point is needed.

This approach was taken by Bandiera (1993), who derived a set of PDEs in curvilinear coordinates matched to the shape of the shell. These equations were then solved under the assumption of radial, constant velocity, isotropic winds from two point sources. The moving stellar wind bow shock problem is formally obtained by taking the limit of one source placed infinitely far away, while allowing its mass loss rate to be infinite so as to produce a finite density near the other star (at the bow shock).

Bandiera noted that for the specific problems cited, the motion within the shell would be along planes. One may readily see this for the bow shock shown in Figure 1. Consider a point on the bow shock surface, and the plane of constant azimuthal angle containing it and the stellar trajectory. The radial wind striking the shell at this point has momentum lying in this plane. Similarly, the ambient medium striking this point has momentum lying in the plane. If the two incident streams mix instantaneously on impact, the resultant momentum must also lie in the same plane. As the fluid flows along the shell, it continues to incorporate momentum contributions lying in the same plane. The collision of radial winds emanating from two point sources gives the same result. Thus, given the perfect mixing assumption, the flow within the shell will lie along planes, a situation we refer to as *meridional flow*. If the shocked fluid does not mix, the fluid trajectories in the shell are not confined to a plane, and further work is needed to solve this more complicated problem.²

The ensuing geometric simplification allowed Bandiera to construct a two-dimensional grid, consisting of neighboring trajectories within the shell at a set of azimuthal angles. His computational method was to integrate the PDEs for mass, normal momentum, and tangential momentum, by marching along the trajectories within the shell as if the equations were ODEs.

²In the case of axisymmetric problems with shear, one may show that for divergent flow fields such as those encountered in bow shocks and colliding winds, the flow geometry and the fluxes of mass, momentum and angular momentum are the same as for the mixed case (Wilkin, Cantó and Raga 2000).

By differencing neighboring solutions, numerical values of the cross-streamline derivatives were obtained and supplied to the ODE integrator. Provided the spacing of the two-dimensional grid is sufficiently fine, his method will yield accurate solutions. Bandiera postulated that the dependence of the equations on derivatives of quantities across streamlines was “fictitious,” although he did not succeed in eliminating it from the equations.

In this contribution, the solution method will be simplified to an integral approach that avoids the need for PDEs, solving purely algebraic equations. Additionally, the solution may conveniently be obtained in terms of ordinary Cartesian, cylindrical polar, or spherical polar coordinates, and does not require a coordinate system matching the shape of the shell.

2.3. The Solution Method

The solution method is based upon the observation that thin shells driven by hypersonic winds are momentum-conserving in the vector sense (Paper I). In order to conserve momentum in steady-state, the momentum flux in the shell must be the *vector sum* of the two incident momentum fluxes, integrated from the standoff point to the point of interest. Such an integration is performed over the area of the shell between two planes of constant azimuthal angle ϕ , in the limit that their separation $\Delta\phi$ is infinitesimal (Fig. 1). For axisymmetric flow with no rotational motion about the symmetry axis, no mass or momentum crosses these bounding planes, so the mass and momentum flowing within such a wedge must exactly equal the sum of the mass or momentum fluxes onto the external faces of the bounding surfaces (shocks). Given the known vector momentum flux, one may determine the shape of the shell, since the fluid must move in the direction of its momentum. While this description is complete, the mathematics was subsequently simplified with the inclusion of an additional, angular momentum flux integral (Paper II). In Papers I and II, only constant velocity, isotropic winds were considered. For such winds, the vector momentum flux incident onto the shocked shell is independent of the detailed shape of the shell, so it is possible to specify the flux of momentum onto the shell from each side analytically. The methods of Papers I and II will now be extended to non-axisymmetric bow shocks and colliding winds, provided the flow is meridional. For meridional flow, the bounding surfaces are again planes of constant ϕ , allowing us to integrate the external fluxes and determine the internal fluxes within the shell. For anisotropic, radial, constant velocity winds, such as those considered by Bandiera, the momentum flux will depend only upon the number of streamlines intersected, and we will be able to obtain solutions analytically.

Proceeding with the solution method, we note the location of the stagnation point, where the two streams collide head-on. For the two-wind collision, this point lies between the two stars, along the line connecting their centers. For the bow shock due to a wind-blowing star moving with respect to the ambient medium, it will lie on the stellar trajectory, in the direction of stellar motion. For either problem, the z-coordinate axis is chosen to contain the stagnation point at $\theta = 0$, with the coordinate origin at the wind source (Fig. 1). In what follows, the wind from

the coordinate origin will be referred to as “the wind”, while the “second wind” may be either a radial wind from a second point source, or the ambient medium in the frame of the moving star, where the ambient velocity is $\mathbf{V}_a = -V_a \hat{\mathbf{z}}$. However, we will show explicit formulas only for the bow shock, reserving detailed treatment of the binary star colliding winds for a future paper. One should bear in mind that the methods shown below will be applicable as well for that problem. Returning to the description of a thin slice of the shell bounded by planes of constant ϕ , the mass, momentum, and angular momentum flux functions per unit azimuthal angle are defined by

$$\Phi_m = \varpi \sigma V_t \sec \eta, \quad (1)$$

$$\Phi = \Phi_m V_t \hat{\mathbf{t}}, \quad (2)$$

$$\Phi_J = \mathbf{R} \times \Phi. \quad (3)$$

Here $\varpi = R \sin \theta$ is the cylindrical radius, and the unit vector at constant ϕ tangent to the shell is $\hat{\mathbf{t}} = \hat{\phi} \times \hat{\mathbf{n}} / |\hat{\phi} \times \hat{\mathbf{n}}|$, where $\hat{\mathbf{n}}$ is the unit outward normal to the shell. Noting that the arc length element ds traced along the shell is given by $(ds)^2 = (dR)^2 + R^2(d\theta)^2 + R^2 \sin^2 \theta (d\phi)^2$, the arc length element traced along the shell at constant θ is $\varpi \sec \eta d\phi$, where the angle η is given by

$$\tan \eta = \frac{\partial R}{\partial \phi} / R \sin \theta. \quad (4)$$

For example, the flux of mass crossing a surface of constant polar angle θ , between the azimuthal angles ϕ and $\phi + d\phi$, is $\Phi_m d\phi$. If V_θ is the θ -component of velocity within the shell, and we write the cylindrical components of the momentum flux function as Φ_ϖ and Φ_z , then these components are related to the angular momentum flux by

$$\Phi_J = \Phi_m R V_\theta \hat{\phi}, \quad (5)$$

$$\Phi_J = R (\Phi_\varpi \cos \theta - \Phi_z \sin \theta) \hat{\phi}. \quad (6)$$

For meridional flow, the angular momentum within the shell will be in the $\hat{\phi}$ -direction, so for the remainder of the discussion we will only need the magnitude of the angular momentum flux $\Phi_J = \Phi_J \cdot \hat{\phi}$.

The mass, momentum, and angular momentum flux functions due to the incident winds are similarly defined. For the first wind, located at the coordinate origin, these are given in terms of the wind density ρ_w and velocity \mathbf{V}_w as

$$\Phi_{m,w} \Delta\phi = \int \int \rho_w (\mathbf{V}_w \cdot \hat{\mathbf{n}}) dA, \quad (7)$$

$$\Phi_w \Delta\phi = \int \int \rho_w (\mathbf{V}_w \cdot \hat{\mathbf{n}}) \mathbf{V}_w dA, \quad (8)$$

$$\Phi_{J,w} \Delta\phi = \int \int \rho_w (\mathbf{V}_w \cdot \hat{\mathbf{n}}) (\mathbf{R} \times \mathbf{V}_w) dA. \quad (9)$$

The area of integration is that between two planes of constant ϕ , from the standoff point to polar angle θ , following the shape of the collision surface, which is determined along the course of the integrations. The sign of the normal direction is chosen so as to point away from the origin. The flux functions for the ambient medium are strictly analogous, except the unit vector normal to the collision surface would be in the reverse direction:

$$\Phi_{m,a}\Delta\phi = - \int \int \rho_a(\mathbf{V}_a \cdot \hat{\mathbf{n}}) dA, \quad (10)$$

$$\Phi_a\Delta\phi = - \int \int \rho_a(\mathbf{V}_a \cdot \hat{\mathbf{n}})\mathbf{V}_a dA, \quad (11)$$

$$\Phi_{J,a}\Delta\phi = - \int \int \rho_a(\mathbf{V}_a \cdot \hat{\mathbf{n}})(\mathbf{R} \times \mathbf{V}_a) dA. \quad (12)$$

If $R(\theta, \phi)$ is the radius of the shell, and its partial derivatives are R_θ and R_ϕ , the surface area element is

$$dA = \sqrt{R^2 + R_\theta^2 + R_\phi^2 \csc^2 \theta} R \sin \theta d\theta d\phi. \quad (13)$$

The unit outward normal $\hat{\mathbf{n}}$ to the surface is given by

$$\hat{\mathbf{n}} = \frac{(R\hat{\mathbf{r}} - R_\theta\hat{\theta} - R_\phi \csc \theta \hat{\phi})}{\sqrt{R^2 + R_\theta^2 + R_\phi^2 \csc^2 \theta}}. \quad (14)$$

Combining these,

$$\hat{\mathbf{n}} dA = (R\hat{\mathbf{r}} - R_\theta\hat{\theta} - R_\phi \csc \theta \hat{\phi}) R \sin \theta d\theta d\phi. \quad (15)$$

For a radial wind, the partial derivatives with respect to θ and ϕ do not enter the expression for $\mathbf{V}_w \cdot \hat{\mathbf{n}} dA$, which is simply

$$\mathbf{V}_w \cdot \hat{\mathbf{n}} dA = V_w R^2 \sin \theta d\theta d\phi. \quad (16)$$

Similarly, for the ambient medium $\mathbf{V}_a = -V_a \hat{\mathbf{z}} = -V_a(\hat{\mathbf{r}} \cos \theta - \hat{\theta} \sin \theta)$, so noting that $d\varpi/d\theta = R \cos \theta + R_\theta \sin \theta$, we have

$$-\mathbf{V}_a \cdot \hat{\mathbf{n}} dA = V_a \varpi d\varpi d\phi. \quad (17)$$

This eliminates some of the partial derivatives that complicated Bandiera's very general treatment of the geometry. The rate at which conserved quantities are advected onto the shell does not depend upon the detailed description of the shell, provided we ensure the correct number of streamlines is counted. Using eq.(16), the resulting forms of eqs.(7-9) for the radial stellar wind are

$$\Phi_{m,w} = \int_0^\theta R^2 \rho_w V_w \sin \theta' d\theta', \quad (18)$$

$$\Phi_w = \int_0^\theta R^2 \rho_w V_w^2 [\hat{\varpi} \sin \theta' + \hat{\mathbf{z}} \cos \theta'] \sin \theta' d\theta', \quad (19)$$

$$\Phi_{J,w} = 0. \quad (20)$$

Here the radial unit vector has been written in terms of its cylindrical polar components, and the integrations are performed at constant ϕ . Because the stellar wind is radial, it imparts no angular momentum to the shell, and the wind angular momentum flux vanishes. Using eq.(17) to simplify eqs.(10-12) for the ambient medium,

$$\Phi_{m,a} = V_a \int_0^\varpi \rho_a \varpi' d\varpi', \quad (21)$$

$$\Phi_a = -V_a^2 \hat{\mathbf{z}} \int_0^\varpi \rho_a \varpi' d\varpi', \quad (22)$$

$$\Phi_{J,a} = V_a^2 \hat{\phi} \int_0^\varpi \rho_a \varpi'^2 d\varpi'. \quad (23)$$

The mass conservation law for the pre-shock radial wind in steady-state is

$$\frac{\partial}{\partial r} \left(r^2 \rho_w V_w \right) = 0. \quad (24)$$

Consequently, if the wind is of mass loss rate \dot{M}_w and streamline-average speed \bar{V}_w , its properties are given by

$$\rho_w V_w = \frac{\dot{M}_w}{4\pi r^2} f_w(\theta, \phi), \quad (25)$$

$$\rho_w V_w^2 = \frac{\dot{M}_w \bar{V}_w}{4\pi r^2} g_w(\theta, \phi). \quad (26)$$

Here the dimensionless functions f_w and g_w are normalized to have unit streamline-average over 4π steradians. Note that eq.(26) assumes the stellar wind to be coasting. Given this assumption, g_w is independent of the radius. The wind mass and momentum fluxes onto the shell may now be written

$$\Phi_{m,w} = \frac{\dot{M}_w}{4\pi} \int_0^\theta f_w(\theta', \phi) \sin \theta' d\theta', \quad (27)$$

$$\Phi_w = \frac{\dot{M}_w \bar{V}_w}{4\pi} \int_0^\theta g_w(\theta', \phi) [\hat{\varpi} \sin \theta' + \hat{\mathbf{z}} \cos \theta'] \sin \theta' d\theta'. \quad (28)$$

In order to obtain steady-state solutions, the ambient density must be independent of the z-coordinate, although its dependence on the remaining ϖ, ϕ coordinates may be arbitrary. Thus, we assume an ambient density of the form

$$\rho_a = \rho_0 f_a(\varpi, \phi), \quad (29)$$

where ρ_0 is the value of the ambient density along the stellar trajectory and $f(0, \phi) = 1$. The final forms of the ambient flux functions are thus

$$\Phi_{m,a} = \rho_0 V_a \int_0^\varpi f_a \varpi' d\varpi'. \quad (30)$$

$$\Phi_a = -V_a \hat{\mathbf{z}} \Phi_{m,a}. \quad (31)$$

$$\Phi_{J,a} = \rho_0 V_a^2 \int_0^\varpi f_a \varpi'^2 d\varpi'. \quad (32)$$

The mass, momentum, and angular momentum flux functions for both the wind and ambient medium are clearly streamline integrals that do not depend upon the detailed shape of the shell - the integration is essentially one over the solid angle of the $\Delta\phi$ wedge, as seen by the origin. This is a consequence of the fact that the pre-shock media conserve these quantities in steady-state.

Suppose we have performed the integrations for the mass, momentum, and angular momentum onto the narrow slice of the shell, for both incident winds, according to eqs.(20,27,28,30-32). In steady-state, these quantities do not accumulate at any location within the shell, but are carried away by the flow within the layer. The conservation laws of mass, momentum, and angular momentum take the form

$$\Phi_m = \Phi_{m,w} + \Phi_{m,a}, \quad (33)$$

$$\Phi = \Phi_w + \Phi_a, \quad (34)$$

$$\Phi_J = \Phi_{J,w} + \Phi_{J,a}. \quad (35)$$

Now we may describe the properties of the shell explicitly in terms of those of the external winds. Equation (6) yields the shell radius

$$R = \frac{\Phi_J}{\Phi_w \cos \theta - \Phi_z \sin \theta}. \quad (36)$$

This equation combines the formulation of Paper II, which was in terms of azimuthally integrated fluxes, with the original treatment of Paper I, using fluxes in an infinitesimally thin wedge. In eq.(36), each momentum flux function within the shell is to be evaluated as the sum of appropriate source terms, using the conservation laws (eqs.[34,35]). It is to be stressed that if the flow is not meridional, eq.(5) does not hold, and the solution method is considerably more complicated. However, eqs.(6,36) are still valid provided the angular momentum flux is replaced by its $\hat{\phi}$ component. From eqs.(1,2), the tangential velocity of matter in the shell is

$$V_t \hat{\mathbf{t}} = \Phi / \Phi_m, \quad (37)$$

while the mass surface density in the shell is given by

$$\sigma = (\Phi_m^2 / \varpi |\Phi|) \cos \eta. \quad (38)$$

2.4. The Applied Torque Method and an Example

To provide a specific example of the solution method, as well as a reference solution to be compared to in the following, we consider the simple problem of a bow shock from an isotropic wind in a uniform ambient medium of density ρ_a . In this case, $f_w = g_w = f_a = 1$, and the flux functions are

$$\Phi_{m,w} = \frac{\dot{M}_w}{4\pi} (1 - \cos \theta), \quad (39)$$

$$\Phi_w = \frac{\dot{M}_w V_w}{8\pi} [\hat{\omega} (\theta - \sin \theta \cos \theta) + \hat{z} \sin^2 \theta], \quad (40)$$

$$\Phi_{J,w} = 0, \quad (20)$$

$$\Phi_{m,a} = \frac{1}{2} \varpi^2 \rho_a V_a, \quad (41)$$

$$\Phi_a = -\frac{1}{2} \varpi^2 \rho_a V_a^2 \hat{z}, \quad (42),$$

$$\Phi_{J,a} = \frac{1}{3} \varpi^3 \rho_a V_a^2. \quad (43)$$

Because the stellar wind has uniform speed, we have replaced \bar{V}_w with V_w . One could immediately substitute these results into eq.(36) to determine the shell's shape. However, before proceeding to apply the formalism, note that the equation for the shape of the shell depends upon a specific combination of three flux functions in eq.(36). One may divide the equation into two parts, one part for each wind, according to

$$\mathcal{T}_k = \varpi[\Phi_{\varpi,k} \cot \theta - \Phi_{z,k}] - \Phi_{J,k}, \quad (44)$$

and $k = w$ or a for the wind and ambient (or second wind) sources. Physically, \mathcal{T} represents the applied torque necessary to compress the fan of streamlines between the stagnation point and position $R(\theta, \phi)$ into a unidirectional beam possessing the same momentum flux (we shall call \mathcal{T} the *required torque*). This is clearly seen by considering the radial wind from the coordinate origin. It has no angular momentum, but an equivalent beam containing the same momentum flux, located at a specific point $R(\theta, \phi)$ does indeed have angular momentum about the origin, because the beam will not be radial. The required torque for the isotropic wind follows from eqs. (20,40,44):

$$\mathcal{T}_w = -\frac{\dot{M}_w V_w \varpi}{8\pi} (1 - \theta \cot \theta). \quad (45)$$

Using eqs.(41-44), the torque necessary to compress the ambient streamlines to a unidirectional beam is

$$\mathcal{T}_a = \frac{1}{6} \varpi^3 \rho_a V_a^2. \quad (46)$$

For the ambient medium, the required torque expression simplifies because the ambient medium has no cylindrical radial momentum, and its z-momentum flux is related to its mass flux by eq.(31), so in general,

$$\mathcal{T}_a = \varpi V_a \Phi_{m,a} - \Phi_{J,a}. \quad (47)$$

Similarly, the radial wind's required torque is simplified due to its lack of angular momentum about the origin.

The solution surface is now defined by

$$\mathcal{T}_w + \mathcal{T}_a = 0. \quad (48)$$

The interpretation of eq.(48) is that each wind supplies the torque necessary to compress the other wind. These torques are equal and opposite only if the correct value of the radius (or cylindrical radius ϖ) is used, conserving angular momentum as well as linear momentum. The utility of this approach is that if we vary the properties of one of the winds, holding the other constant, the shape of the shell is affected only by the change in the applied torque from the second wind.

Now the applied torque expressions for the wind (eq.[45]) and the ambient medium (eq.[46]), when substituted into eq.(48), quickly recover the solution of Paper I,

$$R(\theta) = R_0 \csc \theta \sqrt{3(1 - \theta \cot \theta)}, \quad (49)$$

where R_0 is the standoff radius:

$$R_0 = \sqrt{\frac{\dot{M}_w V_w}{4\pi \rho_a V_a^2}}. \quad (50)$$

3. Solutions for Bow Shock with Ambient Density Gradient

In this section we assume the asymmetry of the bow shock to be due to a density gradient in the ambient medium, while the the stellar wind is assumed to be isotropic. The next section will treat bow shocks from anisotropic winds.

The mass and momentum contributions of the stellar wind to the shell are given by eqs.(39,40,20), and its required torque is given by eq.(45). The ambient medium is assumed to have a plane-parallel density stratification, so in eq.(29), we replace $f_a(\varpi, \phi)$ simply by $f_a(x)$, with f_a is normalized so that $f_a(0) = 1$, and where

$$x = \varpi \cos \phi. \quad (51)$$

The integrated mass and angular momentum flux onto the wedge from the ambient medium are

$$\Phi_{m,a} = \rho_0 V_a \int_0^\varpi f_a(\varpi' \cos \phi) \varpi' d\varpi'. \quad (52)$$

$$\Phi_{J,a} = \rho_0 V_a^2 \int_0^\varpi f_a(\varpi' \cos \phi) \varpi'^2 d\varpi'. \quad (53)$$

A complete description of the bow shock is now obtained by adding the contributions of both the wind and ambient medium using the conservation laws (eqs.[33-35]).

3.1. Solution for Exponential Density Law

We first obtain the solution for an exponential distribution of ambient density. Let the density scale height be H . Define for brevity the coordinate $y = x/H$, so the density law is

$$f_a(x) = \exp(-y), \quad (54)$$

and the mass and angular momentum flux integrals are

$$\Phi_{m,a} = H^2 \rho_0 V_a \sec^2 \phi [1 - \exp(-y)(1 + y)], \quad (55)$$

$$\Phi_{J,a} = H^3 \rho_0 V_a^2 \sec^3 \phi [2 - \exp(-y)(y^2 + 2y + 2)]. \quad (56)$$

The ambient momentum flux follows from eq.(31), while the required torque follows from eq.(47), which yields

$$\mathcal{T}_a = H^3 \rho_0 V_a^2 \sec^3 \phi \left[y - 2 + \exp(-y)(y + 2) \right]. \quad (57)$$

Using this torque formula and that for the stellar wind given by eq.(45), the shell's shape is given by eq.(48), which yields upon simplification

$$l^2 \sec^2 \phi y^{-1} [y - 2 + \exp(-y)(y + 2)] = \frac{1}{2} (1 - \theta \cot \theta). \quad (58)$$

Here $l = H/R_0$ is the density scale height in units of the standoff radius. Letting y_{ax} be the value of $y = x/H$ appropriate for the axisymmetric solution of eq.(49),

$$y_{ax} = \frac{R_0}{H} \cos \phi \sqrt{3(1 - \theta \cot \theta)}, \quad (59)$$

eq.(58) takes the simple form

$$\frac{1}{y} \left[y - 2 + \exp(-y)(y + 2) \right] = \frac{1}{6} y_{ax}^2. \quad (60)$$

This formula may be solved numerically for $y(\theta, \phi)$, so letting y_s denote the solution for y to eq.(60), the shell's shape is given by

$$R(\theta, \phi) = H \sec \phi \csc \theta y_s(\theta, \phi). \quad (61)$$

The result is a family of bow shock solutions distinguished by the value of the nondimensional parameter l . By examining eq.(60) we may deduce several properties of the solution. The left-hand side of the equation is monotonically increasing as a function of y . The signs of y and y_{ax} must be the same, since it is due to the $\cos \phi$ factor, so it follows that y increases monotonically, although in a very non-linear fashion, as a function of y_{ax} . Because the parameter l does not enter eq.(60), we see that there is one universal solution for the problem, although this solution is scaled by the value of R_0 and stretched (distorted) depending upon the value of H/R_0 . The solution for the radius at all angles θ, ϕ and for all values of l follows from Fig. 2. In particular, as $y \rightarrow \infty$, the left hand side of eq.(60) approaches unity, implying a breakout angle for the shell when $y_{ax} = \sqrt{6}$, which yields

$$1 - \theta_\infty \cot \theta_\infty = 2 l^2 \sec^2 \phi. \quad (62)$$

The opening angle θ_∞ depends upon $\cos \phi$, and applies only to $\cos \phi \geq 0$. In the tail of the bow shock, for $\cos \phi < 0$, we have $y \rightarrow -\infty$. In this case, y depends logarithmically upon y_{ax} . The most interesting part of the solution, for $|y| \leq 1$ may be described by an expansion in terms of

small y_{ax} . Noting that for $\cos \phi = 0$ the solution is identical to that of eq.(49), we anticipate that an expansion for small $\cos \phi$ corresponds to an expansion in terms of small y_{ax} about the standard bow shock solution. Letting R_{ax} be the value of $R(\theta)$ for that solution (given by eq.[49]), we obtain

$$R \approx R_{ax} \left[1 + \frac{1}{4} y_{ax} + \frac{13}{160} y_{ax}^2 + \frac{7}{240} y_{ax}^3 + \frac{11843}{1075000} y_{ax}^4 + \dots \right]. \quad (63)$$

The behavior of the solution near the stagnation point is given by

$$R = R_0 \left[1 + \frac{\cos \phi}{4l} \theta + \left(\frac{1}{5} + \frac{13 \cos^2 \phi}{160 l^2} \right) \theta^2 + \dots \right]. \quad (64)$$

Unlike the axisymmetric bow shock, there is a linear term in the behavior near the standoff point, so it is not describable as parabolic with the z coordinate axis as the axis of a parabola, except for the special angles $\phi = \pi/2$ or $3\pi/2$. As a consistency check, note that as the scale height increases relative to the standoff radius ($l = H/R_0 \rightarrow \infty$), the solution reduces to the standard axisymmetric bow shock (Paper I). The lowest order effect near the standoff point is a *tilt* of the bow shock head, described by the linear term in θ . This means that although the wind and ambient streamlines meet head-on at the standoff point, they are not normal to the shell at this location, quite different from the axisymmetric case. This effect is due to the instantaneous mixing we have assumed. If shear is present in the shell, the two colliding flows could have separate stagnation points, which would be located where the incident stream is indeed normal to the shell. These bow shock solutions differ from the standard model (Baranov, Krasnobaeu & Kulikovskii 1971) in that the exponential ambient density distribution implies a finite total mass on one side of the bow shock. The solution then more closely resembles those of two-wind collisions (Paper II) where there is a finite opening angle for the bow shock tail. Similar problems include blast waves in finite mass media (Koo & McKee 1990) and wind breakout from a stratified medium (Cantó 1980; Borkowski, Blondin & Harrington 1997). For the high density region, $\cos \phi < 0$, the shell's asymptotic shape in the tail region is

$$z \approx -\frac{2l^2}{\pi} \varpi \sec^2 \phi \exp\left(-\frac{\varpi}{H} \cos \phi\right). \quad (65)$$

The shell continues to be more distorted as it expands due to the $\exp(-y)$ factor. Because the shell has a finite opening angle on one side, not all wind streamlines intersect the shell, but those with $\theta > \theta_\infty(\phi)$ freely expand to infinity. Further consequences of the finite opening angle of the shell are discussed in §5.

3.2. Solution for Linear or Polynomial Density Law

Consider a stratification of the ambient medium according to

$$f_a(x) = 1 + a_1 x + a_2 x^2 + \dots \quad (66)$$

Of course, one must ensure that this expression is non-negative over the domain of interest. The fluxes of mass and angular momentum onto the shell are

$$\Phi_{m,a} = \varpi^2 \rho_0 V_a \left[\frac{1}{2} + \frac{a_1}{3}x + \frac{a_2}{4}x^2 + \dots \right], \quad (67)$$

$$\Phi_{J,a} = \varpi^3 \rho_0 V_a^2 \left[\frac{1}{3} + \frac{a_1}{4}x + \frac{a_2}{5}x^2 + \dots \right]. \quad (68)$$

The ambient momentum flux follows from eq.(31), while the required torque follows from eq.(47), giving the result

$$\mathcal{T}_a = \frac{\varpi^3}{6} \rho_0 V_a^2 \left[1 + \frac{a_1}{2}x + \frac{3}{10}a_2x^2 + \dots + \frac{6(n+1)!}{(n+3)!}a_nx^n + \dots \right]. \quad (69)$$

Substitution into eq.(48), we obtain upon simplification

$$\frac{\varpi^2}{R_0^2} \left[1 + \frac{a_1}{2}\varpi \cos \phi + \frac{3a_2}{10}\varpi^2 \cos^2 \phi + \dots \right] = 3(1 - \theta \cot \theta). \quad (70)$$

Restricting the treatment to a linear density gradient, with only a_1 non-zero, this equation is cubic in the variable ϖ . One may solve analytically or numerically for the function $\varpi(\theta, \phi)$, so the solution surface is then given by $R(\theta, \phi) = \varpi(\theta, \phi) \csc \theta$. Inclusion of higher order terms in the density law simply increases the degree of the polynomial, but the solution is obtained in the same way.

The behavior of the solution near the stagnation point is given by

$$\frac{R}{R_0} = 1 - \frac{a_1 R_0 \cos \phi}{4} \theta + \left(\frac{1}{5} + \left(\frac{5}{32}a_1^2 - \frac{3}{20}a_2 \right) R_0^2 \cos^2 \phi \right) \theta^2 + \dots \quad (71)$$

As a consistency check of the solutions, for the case of a uniform ambient medium ($a_1, a_2, \dots = 0$), eqs.(70,71) reduce to the standard solution of Paper I. As a further check, note that this result is consistent with the solution for an exponential mass distribution, if we choose the coefficients corresponding to the exponential distribution of the previous subsection, $a_1 = -1/H, a_2 = 1/2H^2, \dots$, recovering eq.(64).

4. Treatment of Anisotropic Winds

4.1. The Axisymmetric Wind

We now relax the assumption that the wind is isotropic to permit an axisymmetric wind, where the axis of symmetry is misaligned with the direction of stellar motion, or in the case of a two-wind collision in a binary star, where the axis does not point to the other star. Treatment of the accelerated wind will be deferred to a future contribution; in this paper the wind is assumed to be coasting. The coordinate axes are chosen so that the z-direction points in the direction of

stellar motion, or towards the second star. In terms of a spherical coordinate system, where the azimuthal angle ϕ is measured about the z-axis, motion of the shocked fluid in the shell is along planes of constant ϕ . We also define starred coordinates so that the z_* -axis is the symmetry axis of the stellar wind. The two coordinate systems are related by

$$\sin \theta_* \cos \phi_* = \sin \theta \cos \phi, \quad (72)$$

$$\sin \theta_* \sin \phi_* = \sin \theta \sin \phi \cos \lambda - \cos \theta \sin \lambda, \quad (73)$$

$$\cos \theta_* = \sin \theta \sin \phi \sin \lambda + \cos \theta \cos \lambda. \quad (74)$$

The wind mass and momentum flux densities depend on the polar angle θ_* :

$$\rho_w V_w = \frac{\dot{M}_w}{4\pi r^2} f_w(\theta_*), \quad (75)$$

$$\rho_w V_w^2 = \frac{\dot{M}_w \bar{V}_w}{4\pi r^2} g_w(\theta_*). \quad (76)$$

The nondimensional functions f_w and g_w are normalized to have unit average value over 4π steradians.

The incident fluxes of mass, momentum, and angular momentum from the wind onto the shell are

$$\Phi_{m,w}(\theta, \phi) = \frac{\dot{M}_w}{4\pi} F_w(\theta, \phi), \quad (77)$$

$$\Phi_w(\theta, \phi) = \frac{\dot{M}_w \bar{V}_w}{4\pi} \mathbf{G}_w(\theta, \phi), \quad (78)$$

$$\Phi_{J,w}(\theta, \phi) = 0. \quad (80)$$

where the nondimensional functions F_w and $\mathbf{G}_w = G_{w,\varpi} \hat{\varpi} + G_{w,z} \hat{\mathbf{z}}$ are given by

$$F_w = \int_0^\theta f_w \sin \theta' d\theta', \quad (79)$$

$$\mathbf{G}_w = \int_0^\theta g_w [\hat{\varpi} \sin \theta' + \hat{\mathbf{z}} \cos \theta'] \sin \theta' d\theta'. \quad (80)$$

Here f_w and g_w have argument $\theta_*(\theta', \phi)$, so the integrations are performed using the transformation eqs.(72-74) to evaluate $\theta_*(\theta, \phi)$. We also define a nondimensional function T_w associated with the required torque \mathcal{T}_w to compress the wind streamlines according to

$$\mathcal{T}_w = \frac{\dot{M}_w \bar{V}_w}{4\pi} \varpi T_w. \quad (81)$$

Because the wind's angular momentum flux vanishes, eq.(44) implies that

$$T_w = G_{w,\varpi} \cot \theta - G_{w,z}. \quad (82)$$

The functions $f_w(\theta_*)$, $g_w(\theta_*)$ may now be expanded in terms of powers of $\cos \theta_*$,

$$f_w(\theta_*) = \sum_{i=0}^{\infty} b_i \cos^i \theta_*, \quad (83)$$

$$g_w(\theta_*) = \sum_{i=0}^{\infty} c_i \cos^i \theta_*. \quad (84)$$

Defining for brevity

$$p = \sin \phi \sin \lambda, \quad (85)$$

$$q = \cos \lambda, \quad (86)$$

the transformation given by eq.(74) is $\cos \theta_* = p \sin \theta + q \cos \theta$. Denoting the trigonometric integrals by

$$I_{j,k} = \int_0^\theta \sin^j \theta' \cos^k \theta' d\theta', \quad (87)$$

we evaluate F_w and \mathbf{G}_w using the above series, and bringing p and q outside of the integrals, we finally have

$$F_w = \sum_{i=0}^{\infty} b_i \sum_{j=0}^i p^{i-j} q^j \binom{i}{j} I_{1+i-j,j}, \quad (88)$$

$$\mathbf{G}_w = \sum_{i=0}^{\infty} c_i \sum_{j=0}^i p^{i-j} q^j \binom{i}{j} [\hat{\omega} I_{2+i-j,j} + \hat{\mathbf{z}} I_{1+i-j,1+j}]. \quad (89)$$

Here we have used the binomial coefficients to write the sums. The applied torque necessary to compress the wind streamlines to a thin shell, in nondimensional form, is now

$$T_w = \sum_{i=0}^{\infty} c_i T_w^{(i)}, \quad (90)$$

where the responses due to the individual $\cos^i \theta_*$ terms are given by

$$T_w^{(i)} = \sum_{j=0}^i p^{i-j} q^j \binom{i}{j} [\cot \theta I_{2+i-j,j} - I_{1+i-j,1+j}]. \quad (91)$$

4.2. General Solution for Bow Shock Driven by an Axisymmetric, Misaligned Wind

The wind description of the previous subsection may now be applied to the problem of a bow shock driven by an anisotropic wind. The wind is assumed to be driven by a star moving at speed V_a in a medium of uniform density ρ_a . We describe the bow shock's properties in the frame of the star.

The standoff radius, defined as the shell radius at $\theta = 0$, which corresponds to $\theta_* = \lambda$, is given by

$$R_\lambda = \sqrt{\frac{\dot{M}_w \bar{V}_w g_w(\lambda)}{4\pi \rho_a V_a^2}}. \quad (92)$$

We also define R_0 to be the standoff radius for the equivalent isotropic wind

$$R_0 = \sqrt{\frac{\dot{M}_w \bar{V}_w}{4\pi \rho_a V_a^2}}. \quad (93)$$

It is important to recall for the results below that the standoff radius will be $R_0 \sqrt{g_w(\lambda)}$, where

$$g_w(\lambda) = \sum_{i=0}^{\infty} c_i \cos^i \lambda. \quad (94)$$

By eqs.(46,48,82), the solution for the shell's radius is

$$R(\theta, \phi) = R_0 \csc \theta \sqrt{-6 T_w}, \quad (95)$$

where T_w is given by eqs.(90,91). The total mass and momentum fluxes, including the contribution from the ambient medium, are

$$\Phi_m = \frac{\dot{M}_w}{4\pi} [F_w + \frac{1}{2\alpha} \tilde{\omega}^2], \quad (96)$$

$$\mathbf{\Phi} = \frac{\dot{M}_w \bar{V}_w}{4\pi} [G_{w,\varpi} \hat{\omega} + (G_{w,z} - \frac{1}{2} \tilde{\omega}^2) \hat{\mathbf{z}}], \quad (97)$$

where $\tilde{\omega} = \varpi/R_0$, and $\alpha = V_a/\bar{V}_w$.

To obtain a complete description of the bow shock's properties, we must specify the velocity of material in the shell and the mass surface density of matter. The velocity is given by the ratio of the mass and momentum fluxes:

$$V_t \hat{\mathbf{t}} = V_a \frac{[2 G_{w,\varpi} \hat{\omega} + (2 G_{w,z} - \tilde{\omega}^2) \hat{\mathbf{z}}]}{[2 \alpha F_w + \tilde{\omega}^2]}. \quad (98)$$

The surface density is given by eq.(38), which yields

$$\sigma = \rho_a R_0 \frac{[2 \alpha F_w + \tilde{\omega}^2]^2}{2 \tilde{\omega} \sqrt{[4 G_{w,\varpi}^2 + (2 G_{w,z} - \tilde{\omega}^2)^2]}} \cos \eta. \quad (99)$$

In order to evaluate σ , we need to know the angle η . Using eqs.(4,95), we have

$$\tan \eta = \frac{1}{2} \csc \theta \frac{1}{T_w} \frac{\partial T_w}{\partial \phi}, \quad (100)$$

where differentiation of eqs.(90,91) with respect to ϕ , which enters only in the variable p , yields

$$\frac{\partial T_w}{\partial \phi} = \sum_{i=1}^{\infty} c_i \frac{\partial T_w^{(i)}}{\partial \phi}, \quad (101)$$

$$\frac{\partial T_w^{(i)}}{\partial \phi} = \cot \phi \sum_{j=0}^{i-1} (i-j) p^{i-j} q^j \binom{i}{j} [\cot \theta I_{2+i-j,j} - I_{1+i-j,1+j}]. \quad (102)$$

Note that the summation limits are restricted so that the terms with $j = i$ vanish, including the c_0 term.

In the region of the standoff point, an expansion to second order in θ gives

$$R \approx R_\lambda \left\{ 1 + \theta \frac{p g'_w(\lambda)}{4 g_w(\lambda)} + \theta^2 \left[\frac{1}{5} - \frac{p^2 g_w'^2(\lambda)}{32 g_w^2(\lambda)} + \frac{3(p^2 g_w''(\lambda) - q g'_w(\lambda))}{40 g_w(\lambda)} \right] \right\}. \quad (103)$$

Primes on the function g_w represent differentiation with respect to $q = \cos \lambda$. As was the case of asymmetry resulting from an ambient density gradient, there is a θ^1 term indicating a tilt of the standoff region relative to the direction of the stellar motion. The mass and momentum flux functions near the standoff point are

$$\Phi_m \approx \frac{\dot{M}_w}{4\pi} \left\{ \frac{1}{2} f_w(\lambda) + \frac{1}{2\alpha} g_w(\lambda) \right\} \theta^2. \quad (104)$$

$$\Phi \approx \frac{\dot{M}_w \bar{V}_w}{4\pi} \left\{ \frac{1}{3} g_w(\lambda) \hat{\omega} + \frac{p}{3} g'_w(\lambda) \hat{\mathbf{z}} \right\} \theta^3. \quad (105)$$

Equations (4,28,29) now give the tangential velocity in the shell and the mass per unit area as

$$V_t \hat{\mathbf{t}} \approx \frac{2}{3} V_a \theta \frac{g_w(\lambda) \hat{\omega} + p g'_w(\lambda) \hat{\mathbf{z}}}{\alpha f_w(\lambda) + g_w(\lambda)}, \quad (106)$$

and

$$\sigma \approx \frac{3 R_0 \rho_a}{4} \frac{(\alpha f_w(\lambda) + g_w(\lambda))^2}{\sqrt{g_w(\lambda) (g_w^2(\lambda) + p^2 (g'_w(\lambda))^2)}}. \quad (107)$$

One may readily see that these solutions reduce to the results of Paper I for an isotropic wind ($f_w = g_w = 1$).

4.3. Simple Solutions: Quadratic Dependence on $\cos \theta_*$

The simplest non-trivial solution is for a linear dependence of mass and momentum fluxes upon $\cos \theta_*$. However, in astrophysical applications one is frequently concerned with stellar winds that have symmetry with respect to the equatorial plane, such as when the asymmetry arises due to rotation of the star. For winds that are symmetric with respect to $\theta = \pi/2$, the functions $f_w(\theta_*)$, $g_w(\theta_*)$ will have expansions in terms of even powers of $\cos \theta_*$ only. Thus, we shall give the solution for a wind that depends upon $\cos^2 \theta_*$. This is sufficiently general to include as a subset

previous models of non-axisymmetric bow shocks (Bandiera 1993; Chen & Huang 1997). We assume the mass and momentum fluxes are described by

$$f_w = b_0 + b_1 \cos \theta_* + b_2 \cos^2 \theta_*, \quad (108)$$

$$g_w = c_0 + c_1 \cos \theta_* + c_2 \cos^2 \theta_*, \quad (109)$$

where the normalization requires $b_0 = (1 - b_2/3)$ and $c_0 = (1 - c_2/3)$.

The mass flux integral F_w is

$$F_w = \left\{ b'_0 (1 - \mu) + \frac{b_1}{2} [p (\theta - \sin \theta \cos \theta) + q \sin^2 \theta] \right. \\ \left. + \frac{b_2}{3} \sin^2 \theta [(q^2 - p^2) \cos \theta + 2pq \sin \theta] \right\}, \quad (110)$$

where $b'_0 = b_0 + b_2(2p^2 + q^2)/3$. The components of \mathbf{G}_w are

$$G_{w,\varpi} = \frac{1}{4} \left\{ c'_0 (2\theta - \sin 2\theta) + \frac{c_1}{3} [p (8 - 9 \cos \theta + \cos 3\theta) + 4q \sin^3 \theta] \right. \\ \left. + c_2 \sin^3 \theta [2pq \sin \theta + (q^2 - p^2) \cos \theta] \right\}, \quad (111)$$

$$G_{w,z} = \frac{1}{4} \left\{ c'_0 (1 - \cos 2\theta) + \frac{c_1}{3} [4p \sin^3 \theta + q (4 - 3 \cos \theta - \cos 3\theta)] \right. \\ \left. + c_2 \left[\frac{pq}{4} (4\theta - \sin 4\theta) + \frac{(q^2 - p^2)}{2} (2 + \cos 2\theta) \sin^2 \theta \right] \right\}. \quad (112)$$

Here the coefficient $c'_0 = c_0 + c_2(3p^2 + q^2)/4$ depends on ϕ and λ . Using eqs.(82,111,112) we obtain

$$T_w = \left\{ -\frac{c'_0}{2} (1 - \theta \cot \theta) - \frac{c_1}{3} (1 - \cos \theta) [q + p \tan(\frac{\theta}{2})] \right. \\ \left. + \frac{c_2}{8} [(p^2 - q^2) \sin^2 \theta - pq(2\theta - \sin 2\theta)] \right\}. \quad (113)$$

Using eq.(95), with T_w given by eq.(113), we obtain the shell's shape

$$R = R_0 \csc \theta \left\{ 3(1 - \theta \cot \theta) (c_0 + \frac{c_2}{4} (3p^2 + q^2)) \right. \\ \left. + 2c_1 (1 - \cos \theta) [q + p \tan(\frac{\theta}{2})] \right. \\ \left. + \frac{3c_2}{4} [(q^2 - p^2) \sin^2 \theta + pq(2\theta - \sin 2\theta)] \right\}^{1/2}, \quad (114)$$

where the explicit dependence upon stellar inclination and azimuthal angle is obtained using $p = \sin \phi \sin \lambda$ and $q = \cos \lambda$. Examples are shown in Figures 4-6 for even parity winds with

$c_2 = 3, 1.5$, and -1.5 . Noting that Bandiera’s notation for this problem was $\Delta_Y = -2c_2/3$, these correspond to his cases of $\Delta_Y = -2, -1$ and $+1$.

In the neighborhood of the standoff point, the shell’s shape is given by

$$R^2/R_0^2 \approx [(c_0 + c_1 q + c_2 q^2) + \frac{p}{2}(c_1 + 2c_2 q)\theta + (8c_0 + 5c_1 q + 2c_2(3p^2 + q^2))\theta^2/20]. \quad (115)$$

4.4. The Non-Axisymmetric Wind

The previous formulation is not restricted to an axisymmetric wind, so one may obtain solutions for bow shocks driven by radial, non-axisymmetric winds, by defining the momentum flux from the wind in terms of spherical harmonics. Because the method is clear from the previous discussion of an axisymmetric wind, we will not give further details here. We note only that the response from each spherical harmonic contribution to the wind momentum flux may be obtained from the required torque formula. Then the shape of the shell will depend upon the square root of the sum of these responses.

5. Energy Thermalization Rates

The energy thermalization rate of axisymmetric bow shocks and colliding winds has been considered by Wilkin, Cantó & Raga (2000). The thermalization occurs in the shocks, post-shock (radiative) relaxation, and mixing. The maximum possible rate of thermalization is given by the total incident kinetic energy flux in the center of mass frame. Here we are concerned with the total rate of energy thermalized, over the entire shell, rather than the amount per unit area. By center of mass, we mean the center of mass of the matter deposited onto the shell per unit time. This maximum thermalization rate is for the case of complete mixing. For the bow shock in a uniform ambient medium, the center of mass frame corresponds to the ambient frame, because a formally infinite amount of ambient mass per unit time strikes the shell, while the stellar wind has finite mass loss rate. For the bow shock driven by a star with a radial wind, the total energy thermalization rate is given by

$$\dot{E}_{\text{therm}} = \int_0^{2\pi} \int_0^\pi \frac{1}{2} R^2 \rho_w V_w'^2 (\mathbf{V}_w' - \mathbf{V}_{\text{shell}}') \cdot \hat{\mathbf{r}} \sin \theta d\theta d\phi. \quad (116)$$

Here primes indicate velocities in the ambient frame, so $\mathbf{V}_w' = V_w \hat{\mathbf{r}} + V_a \hat{\mathbf{z}}$, giving

$$V_w'^2 = [V_w^2 + 2V_w V_a \cos \theta + V_a^2]. \quad (117)$$

The velocity of the shell is its pattern speed and corresponds to the stellar velocity $\mathbf{V}_* = V_a \hat{\mathbf{z}}$. Because we are assuming a non-accelerating wind, the integral depends only upon the streamlines and not the detailed shape of the shell, so we may perform the integration over a spherical surface.

For the case of an isotropic wind from a star moving through a plane-parallel stratified ambient medium, the rate of energy thermalization is precisely the same as for a uniform medium (Wilkin, Cantó, & Raga 1997), since it is the kinetic energy in the ambient frame that is thermalized:

$$\dot{E}_{\text{therm}} = \frac{1}{2} \dot{M}_w (V_w^2 + V_*^2). \quad (118)$$

This rate of energy thermalization is appropriate for an arbitrary ambient distribution of matter, provided only that it is truly infinite in mass, and that the bow shock is steady-state in the star's frame. The second requirement demands that the ambient mass distribution be independent of the z -coordinate, although its (ϖ, ϕ) distribution is arbitrary. An exception, for example, is the case of an exponential stratification, for which case the total ambient mass flux may be finite (on one side of the bow shock). For this case, the center of mass frame is not equal to the ambient frame, and it is important to separately define the center of mass frame for each ϕ slice, as it will depend upon azimuthal angle for the portion of the bow shock that contains finite opening angle. Because of the resulting opening angle, not all of the stellar wind kinetic energy (in the ambient frame) will be thermalized, for two reasons: (1) some streamlines miss the shell, and (2) there is a net flow of momentum along the shell even at infinite distance, whereas for the standard bow shock $V_t \rightarrow 0$ in the ambient frame, far in the bow shock tail.

We now consider the case of an anisotropic wind, to determine whether the energy thermalization rate depends on the orientation of the wind. However, the thermalization rate remains independent of the stratification of the ambient medium, subject to the caveat about finite-mass systems. For the remainder of this section, we allow the wind to be non-axisymmetric, so in equations (75,76) we use $f_w = f_w(\theta_*, \phi_*)$, $g_w = g_w(\theta_*, \phi_*)$. When we refer to an axisymmetric wind, its axis of symmetry will be $\theta_* = 0$ as before.

The total kinetic energy loss rate of the wind is

$$\dot{E}_w = \frac{1}{8\pi} \dot{M}_w \bar{V}_w^2 \int_0^{2\pi} \int_0^\pi \frac{g_w^2}{f_w} \sin \theta_* d\theta_* d\phi_*. \quad (119)$$

The functions f_w and g_w are assumed to have unit average over 4π steradians. In terms of this energy loss rate, we define the mean square wind speed as

$$\langle V_w^2 \rangle = 2\dot{E}_w / \dot{M}_w. \quad (120)$$

We also define a *total* vector momentum flux for the wind, a quantity that is non-vanishing only if the wind is not point-symmetric with respect to the origin

$$\mathbf{P}_w = \frac{\dot{M}_w \bar{V}_w}{4\pi} \int_0^{2\pi} \int_0^\pi g_w \hat{r} \sin \theta d\theta d\phi, \quad (121)$$

$$= \int_0^{2\pi} \mathbf{\Phi}_w(\pi, \phi) d\phi. \quad (122)$$

We now wish to calculate the kinetic energy deposition rate to the shell in the ambient frame. Letting $\alpha = V_*/\bar{V}_w$, the wind velocity (squared) in the ambient frame becomes

$$V_w'^2 = \bar{V}_w^2 \left[\frac{g_w^2}{f_w^2} + 2\alpha \frac{g_w}{f_w} \cos \theta + \alpha^2 \right]. \quad (123)$$

The wind density is given by

$$4\pi R^2 \rho_w = \frac{\dot{M}_w f_w^2}{\bar{V}_w g_w}. \quad (124)$$

The total kinetic energy flux onto the shell in the ambient frame is

$$\dot{E}_{\text{therm}} = \frac{\dot{M}_w \bar{V}_w^2}{8\pi} \int_0^{2\pi} \int_0^\pi \left[\frac{g_w^2}{f_w^2} + 2\alpha g_w \cos \theta + \alpha^2 f_w \right] \sin \theta_* d\theta_* d\phi_*. \quad (125)$$

The normalization condition for f permits the last term to be easily integrated. The second term is V_* times the z component of the wind momentum flux. In terms of the total wind kinetic energy flux and momentum flux, we have

$$\dot{E}_{\text{therm}} = \frac{1}{2} \dot{M}_w (< V_w^2 > + V_*^2) + \mathbf{P}_w \cdot \mathbf{V}_*. \quad (126)$$

We see that a sufficient condition for the total thermalization rate to be independent of the wind orientation is the vanishing of $P_{z,w}$. Because we do not wish to invoke a special alignment of the wind orientation with the star's direction of motion, more typically this requires a point-symmetric wind such that $\mathbf{P}_w = 0$. As a special case, this can be more explicitly confirmed for the axisymmetric wind, assuming it is symmetric with respect to its midplane $\theta_* = \pi/2$. Because we are considering the total thermalization rate, we may perform the solid angle integration in starred coordinates. In this case, we need only the transformation equation for $\cos \theta$ which is given by eq.(74). We assume the wind to be axisymmetric and symmetric with respect to $\theta_* = \pi/2$, so it has an expansion in $\cos \theta_*$ with only even powers. This implies that the $2\alpha \cos \theta$ term in the integral will give no contribution, by parity, because it yields only a $\sin \phi_*$ term, which vanishes in the azimuthal integration and terms with odd powers of $\cos \theta_*$, which vanish in the polar angle integration. The remaining contribution is then

$$\dot{E}_{\text{therm}} = \frac{\dot{M}_w \bar{V}_w}{4} \int_0^\pi \left[\frac{g_w^2(\theta_*)}{f_w(\theta_*)} + \alpha^2 f_w(\theta_*) \right] \sin \theta_* d\theta_*, \quad (127)$$

$$= \frac{1}{2} \dot{M}_w (< V_w^2 > + V_*^2). \quad (128)$$

The total energy thermalization rate is independent of the orientation of the midplane-symmetric, axisymmetric wind. For the case of a non-point-symmetric wind, including an axisymmetric wind that doesn't possess mirror symmetry with respect to the equator, the total thermalization rate will depend upon the stellar orientation because of the contribution from the $2\alpha \cos \theta$ term. This

is the case of the example solution given in Section 4.3, for the wind with quadratic dependence on $\cos \theta_*$. For that wind, the total vector momentum loss rate is

$$\mathbf{P}_w = \frac{\dot{M}_w \bar{V}_w}{3} c_1 \hat{\mathbf{z}}_*, \quad (129)$$

where $\hat{\mathbf{z}}_*$ points in the direction of the wind symmetry axis z_* . The thermalization rate for the bow shock driven by this wind therefor has a fluctuation depending on orientation of the wind symmetry axis with respect to the stellar velocity of magnitude

$$\Delta \dot{E}_{therm} = \frac{\dot{M}_w \bar{V}_w V_*}{3} c_1 \cos \lambda. \quad (130)$$

If the quadratic term in the wind momentum vanishes ($c_2 = 0$), then the absolute value of c_1 may not exceed unity. The amount this changes the total thermalization rate then depends upon the parameter $\alpha = V_*/V_w$.

We must note that changing orientation of the point-symmetric wind did not change the total thermalization rate, but the spectrum of shock emission will clearly be different. For example the peak post-shock temperature will depend upon the incident normal component of velocity. The sum of thermalization by shock, relaxation, and mixing is independent of orientation, but the individual contributions will vary. This highlights the fact that the properties of bow shocks due to non-axisymmetric winds may be determined by measuring the bow shock shape, mass surface density (or column density), kinematics, and radiated energy. Although detailed shock calculations are beyond the scope of this work, a substantial amount of information should be obtainable from these global quantities and may be sufficient for sources where the data are sparse.

6. Summary

I have shown how to solve the problem of non-axisymmetric bow shocks and wind collisions with a simple formalism that requires only algebraic equations. Most often one considers constant wind speed for such problems, and in this case the method leads to exact, analytic (although possibly implicit) solutions. The availability of simple analytic solutions makes it much easier to model observed sources and derive the properties of the driving winds. Among the principal applications of these solutions would be to determine the cause of the asymmetry in observed bow shocks, whether it be due to the ambient medium or an anisotropic wind, or both.

Future improvements are needed to include non-radial and accelerating winds and shearing motions in the shell, in which case the fluid elements are not restricted to a plane. Also, non-axisymmetric bow shocks due to colliding winds in binary systems, including the orbital motion, require a formulation in a non-inertial frame. A forthcoming paper will describe how to solve the general problem of colliding winds from two stars, including both anisotropy and acceleration in the winds.

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Fig. 1.— Thin Shell Model for a Stellar Wind Bow Shock. The top panel defines the spherical coordinate system and shows a thin wedge cut by two planes of constant ϕ , while the bottom panel shows the wind, ambient, and tangential flows in the frame of the star.

Fig. 2.— Solution of eq.(60): Scaled x-coordinate versus that of a bow shock in a uniform medium.

Fig. 3.— Solution surfaces for $l = 3$ (top) and $l = 1$ (bottom). Stars mark the source of the wind. The ambient density increases towards the left of the page, and the direction of stellar motion is downwards. The bow shocks are viewed from an angle of 10° by rotating about the x-axis. Contours of constant θ are at every 8° up to 120° , while contours of constant ϕ are at every 10° .

Fig. 4.— Bow Shock for $c_2 = 3$, a polar wind. The wind is inclined with respect to the z-direction by angles (left column, from top to bottom) 0,10,30 degrees, and (right column, top to bottom) by 50,70, and 90 degrees.

Fig. 5.— Bow Shock for $c_2 = 1.5$, a polar wind. The wind is inclined by angles 0,10,30,50,70, and 90 degrees with respect to the z-direction as in Figure 4.

Fig. 6.— Bow Shock for $c_2 = -1.5$, an equatorial wind. The wind is inclined by angles 0,10,30,50,70, and 90 degrees with respect to the z-direction as in Figure 4.